Investment and hiring behaviors with fixed and proportional costs

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Abstract

A firm’s stochastic dynamic of investment and employment decisions are examined when investment is irreversible and accompanied by a fixed cost and labor is subject to “kinked” linear adjustment cost. This framework allows us to fully characterize the optimal interrelated dynamics of hiring, firing and investment in a case where the Cobb-Douglas technology is perturbed by a geometric brownian motion. Both adjustments are lumpy and asymmetric in the sense that an investment spike is necessarily accompanied by a hiring spike, while labor adjustments may occur without investment. We determine the ergodic set in which firms are trapped in the long-run and the associated distribution of the relevant variables over this set.

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1 Introduction

As it has been emphasized by Doms and Dunne [1998] for US, or by Duhautois and Jamet [2002] for France, investment at the firm level is lumpy and gives rise to sudden spike. This fact may reflect the existence of lump-sum cost of adjustment of capital which impedes smooth investment patterns.

Recent empirical papers by Letterie, Pfann, and Polder [2004] or by Sakellaris [2001] shows that these investment spikes are often accompanied by active hiring or firing episodes. This interrelation between capital and labor adjustments have two sources: first, marginal return of a factor is necessarily influenced by the level of the other factor, second, adjustment costs can be interrelated in the sense of Nadiri and Rosen [1969].

This paper provides a stochastic framework in which irreversible capital adjustment entails a lump sum cost, while asymmetric hiring and firing linear costs shapes the adjustment of the labor force. Due to the presence of the fixed cost, capital and labor may have simultaneous positive jumps at isolated date, while the firm may continuously regulate its labor force as in Bentolila and Bertola [1990].

This singular dynamics distinguishes our paper from Dixit [1997], Eberly and Van Mieghem [1997] or Abel and Eberly [1998]. Dixit [1997] or Eberly and Van Mieghem [1997] consider a case in which capital and labor are both subject to asymmetric linear adjustment costs. It excludes the occurrence of discrete jumps affecting labor and capital. Abel and Eberly [1998] assume that labor is perfectly flexible while capital entails a lump sum adjustment cost. As labor and capital are complements, a discrete adjustment of capital necessarily implies a simultaneous jump affecting labor. However, this model excludes situation in which lumpy episodes of hiring or firing occurs independently of investment spikes.

Our framework permits to emphasize the link between investment policy and hiring/firing behavior. What are the consequences of hiring and firing costs on the return to capital?

The second section introduces the framework and presents the dynamic problem solved by a firm. The third section provides a recursive formulation by giving a set of properties satisfied by the value function associated to the firm’s problem. Section 4 characterizes the ergodic set and derives the stationary distribution among firms of the relevant variable.

2 The firm’s problem

2.1 Technology and structure of the adjustment costs

We consider a firm with a technology, that uses labor, \( L \), and capital \( K \) as factors of production. For simplicity, it is assumed that the wage rate, \( w \), is constant. The profit function is given by:

\[
\Pi(Z, K, L) = Z^{1-a-b}K^aL^b - wL, \quad a, b \in (0, 1), \quad a + b < 1. \tag{1}
\]

\( \Pi \) is a stochastic shock who follows a geometric brownian motion

\[
Z = \{Z(t), t \geq 0\}: \quad dZ(t) = \sigma Z(t)dW, \quad \sigma > 0, \quad t \geq 0 \tag{2}
\]

where \( W \) is a standard Wiener process, with independent, normally distributed increments.

This profit function is consistent with either a price-taking firm whose stochastic production technology exhibits decreasing returns to scale, or with a monopolist facing a constant returns to scale technology and a constant elasticity demand curve that is subject to multiplicative shocks. In any case, a positive shock affecting \( Z \) \((dW > 0)\) has to be seen as a “good” shock, while a negative one is “bad”.

The firm bears capital and labor adjustment costs when it decides to modify the operating input combination \((K, L)\). Capital investment is assumed to be irreversible, while a strictly positive instantaneous investment made at \( t \), causing a jump of capital stock from \( K(t^-) \) to \( K(t^+) \), costs:

\[
cZ(t) + K(t^+) - K(t^-), \quad c > 0.\]
The first component corresponds to a fixed cost, which does not depend on the level of investment. This component is paid as soon as the instantaneous investment is nonzero. As usual, to help long-run consistency, we specify the fixed cost to be proportional to the geometric Brownian motion $Z$. The second component results from the purchase of capital on a competitive market at a unitary constant price.

Concerning labor, there is no fixed cost of adjustment. However, labor is not perfectly flexible and the firm bears a hiring cost, $h > 0$ per new employee, and a firing cost $f > 0$, per dismissed worker. This implies that a labor change at date $t$ from $L(t^-)$ to $L(t^+)$ costs:

$$h \times (L(t^+) - L(t^-))1_{[L(t^+) > L(t^-)]} + f \times (L(t^-) - L(t^+))1_{[L(t^+) < L(t^-)]},$$

where $1_{[\cdot]}$ is the indicator function. Irreversibility on employment would correspond to the case in which $f = \infty$.

If no adjustment is made, $K$ and $L$ depreciate at the same exogenous rate $\delta > 0$. This approximates a situation in which every current employee quits voluntarily the firm with a probability $1 - e^{-\delta}$ in a unit of time. It is assumed that $\delta > \sigma^2$.

The firm is risk-neutral and chooses an employment and investment policy to maximize its expected present value of profits over the infinite future. The discount rate $r$ is constant.

### 2.2 Combined impulse-instantaneous control policy

The optimal policy combines features coming from both instantaneous control problem and impulse control one as introduced by Harrison, Sellke, and Taylor [1983] and Harrison and Taksar [1983]. A **combined impulse-instantaneous control (CIC)** policy $P$ is defined as a triplet of stochastic processes $X = \{X(t), t \geq 0\}$, $H = \{H(t), t \geq 0\}$ and $F = \{F(t), t \geq 0\}$ that are non-decreasing and nonanticipating with respect to the Brownian motion $Z$. $X(t)$, $H(t)$ and $F(t)$ correspond to the cumulative amounts of gross investment, hiring and firing, respectively, effected by the firm over the time interval $[0, t]$.

An essential feature of the processes $X$, $H$ and $F$ lies in their discontinuity. Due to the presence of a fixed cost, capital and also labor may have jumps at time 0 and at positive dates. This means that the problem faced by the firm incorporates “impulse control”. Indeed, concerning capital, we rule out the possibility of “instantaneous control”, whereby the firm continuously regulates the capital stock. With a fixed cost, such a policy leads the firm to incur an infinite costs in finite time.

The process $X$ has to jump at isolated points in time $\{\tau_0 = 0, \tau_1, \tau_2, \ldots\}$ and is constant between these points, i.e.:

$$X(t) = \sum_{\tau_i \leq t} I_{\tau_i},$$

with $I_{\tau_i}$ the strictly positive investment made at time $\tau_i, i = 0, 1, \ldots, \infty$.

The capital stock, starting from $K$, evolve according to:

$$K(t) = e^{-\delta t}K + \sum_{\tau_i \leq t} e^{-\delta(t-\tau_i)}I_{\tau_i}, t \geq 0.$$  \hspace{1cm} (3)

Contrary to capital, there is no fixed cost affecting labor adjustment, so that the firm may regulate continuously its labor force, i.e. exert an instantaneous control. This is the case when the marginal productivity of labor evolves continuously, hitting by the productivity or demand shock $Z(t)$. However, due to the discrete adjustment of capital at time $\tau_i, i = 0, 1, \ldots, \infty$, both $H$ and $F$ may jump at these dates. Let $\Delta H_{\tau_i}$ and $\Delta F_{\tau_i}$ denote these jumps. It will be convenient to introduce $H_c$ and $F_c$ the continuous parts of $H$ and $F$ respectively, meaning that:

$$H_c(t) = H(t) - \sum_{\tau_i \leq t} \Delta H_{\tau_i}, \text{ and } F_c(t) = F(t) - \sum_{\tau_i \leq t} \Delta F_{\tau_i},$$  \hspace{1cm} (4)

\[1\] The assumption of an identical depreciation rate affecting labor and capital is made for analytical convenience.
for all \( t \geq 0 \). Obviously, the processes \( H_c \) and \( F_c \) are continuous and nondecreasing. However, as it is the case in a problem of instantaneous control, these processes without jumps are singular, meaning that they increases at an infinite rate only on a time set that has total measure zero, being a collection of distinct dates.

Using the decomposition (\( \Pi \)), the labor force, starting from \( L \), evolve according to:

\[
L(t) = e^{-\delta t} L + \int_0^t e^{-\delta(t-\tau)} (dH_c(\tau) + dF_c(\tau)) + \sum_{\tau_i \leq t} e^{-\delta(t-\tau_i)} (\Delta H_{\tau_i} + \Delta F_{\tau_i}). \tag{5}
\]

Note that the sample path of \( \{H_c(t)\} \) and \( \{F_c(t)\} \) fail to be differentiable, so that we need to use the Riemann-Stieltjes integral with the sample paths of \( \{H_c(t)\} \) and \( \{F_c(t)\} \) as integrating functions.

We are now ready to give the value of a CIIC policy \( P = \{X, H, F\} \) starting from an initial state \((Z, K, L)\):

\[
\mathcal{V}(P; Z, K, L) = E_Z \left\{ \int_0^\infty e^{-rt} [\Pi(Z(t), K(t), L(t)) - (hdH_c(t) + fdF_c(t))] dt \right\}
+ E_Z \left\{ \sum_{i=0}^\infty e^{-r \tau_i} (cz(\tau_i) + I_{\tau_i}) + \Delta H_{\tau_i} + \Delta F_{\tau_i} \right\},
\]

where \( Z(t), K(t), L(t) \) evolves according to (2) (starting from \( Z \), \( K \) and \( L \)).

One may use the homogeneity property of our framework to reduce its dimensionality. The value function associated with a policy \( P \) is homogeneous of degree 1 with respect to \((Z, K, L)\). For each date \( t \), we define the transformed variables \( k(t) = K(t)/Z(t) \), \( l(t) = L(t)/Z(t) \) and \( z(t) = Z(t)/Z \). The transformed policy is \( p = P/Z \), where equals means that the processes \( X, H, F \) are normalized by \( Z \), the initial value of \( Z \), at each date. We use minuscule characters to denote the normalized process, while \( \pi(\cdot) \) denotes the “intensive” income function: \( \Pi(Z(t), K(t), L(t)) = \pi(K/Z, L/Z) - w(L/Z) \).

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The value associated to a policy is written in intensive terms, that is: \( \pi(p; k, l) = \mathcal{V}(P; 1, k, l) \), so that \( \mathcal{V}(P; Z, K, L) = Z \pi(p; k, l) \). One may check that the “intensive” policy \( p \), starting from \((k, l)\) yields a value:

\[
\pi(p; k, l) = E \left\{ \int_0^\infty e^{-rt} [z(t)(\pi(k(t), l(t)) - w(t)) dt - h dH_c(t) - f dF_c(t)] \right\}
+ E \left\{ \sum_{j=1}^\infty e^{-r \tau_j} (cz(\tau_j) + l_{\tau_j}) + h \Delta H_{\tau_j} + f \Delta F_{\tau_j} \right\},
\]

with the following laws of motion driving the processes \( k \) and \( l \):

\[
z(t)k(t) = e^{-\delta t} k + \sum_{\tau_i \leq t} e^{-\delta(t-\tau_i)} i_{\tau_i},
\]

\[
z(t)l(t) = e^{-\delta t} l + \int_0^t e^{-\delta(t-\tau)} (dH_c(\tau) + dF_c(\tau)) + \sum_{\tau_i \leq t} e^{-\delta(t-\tau_i)} (\Delta h_{\tau_i} + \Delta f_{\tau_i}).
\]

In the absence of control, \( k(t) \) and \( l(t) \) evolve as geometric brownian motions driven by the Wiener process \( W \) with drift \( \sigma^2 - \delta < 0 \):

\[
dl_u(t) = -(\delta - \sigma^2) l_u(t) dt - \sigma l_u(t) dW,
\]

\[
dk_u(t) = -(\delta - \sigma^2) k_u(t) dt - \sigma k_u(t) dW,
\]

where the index \( u \) is for “uncontrolled”. The control problem faced by the firm is now written in terms of the two processes \( k, l \) controlled by the CIIC policy \( p = \{x, h, f\} \), and evolving in fixed proportion when uncontrolled. Among these policies, the optimal policy in intensive terms is noted \( p^* \).
3 A recursive formulation

We use a recursive formulation of our problem dictated by a value function. We introduce $v(k, l)$ as a “candidate” value function, which satisfies various properties and gives the value of the optimal CIIC policy, i.e. $v(k, l) = \pi^*(p^*; k, l)$. This section provides a graphical description of the pattern of adjustment, and a set of properties satisfied by the candidate value function.

3.1 Dividing the $(k, l)$-plane and conjecturing the pattern of adjustment

We begin by defining a “forbidden” region $A$ of the $(k, l)$-plane where the firm cannot stay for any positive amount of time. It means that if the firm starts in or reaches this region, it will immediately be thrown out this area by adjusting both factors or only one. As a consequence, the firm will evolve for all strictly positive dates outside the region $A$.

The complement of this region defines the so-called inaction region $I$. When $(k, l) \in I$, no action is undertaken by the firm, and $(k, l)$ evolves stochastically along a ray emanated from the origin. This means that the ratio $y = k/l$ remains constant in the inaction region. When a positive (“good”) shock $(dW > 0)$ affecting the level of productivity $z$ occurs, the firm moves toward the origin along a ray with slope $y$.

Given the “candidate” value function $v(k, l)$, the inaction region $I$ is defined by the set of $(k, l)$ so that:

$$v(k, l) > \sup_{k' > k, l'} \{ v(k', l') - c1_{[k' > k]} - (k' - k) - h(l' - l)1_{[l' \geq 0]} - f(l - l')1_{[l' < 0]} \}.$$  

It is crucial to characterize the boundaries separating the inaction region from the forbidden one, since a control is exerted by the firm each time these boundaries are attained.

The forbidden region $A$ is divided in two disjoint subsets $A_1$ and $A_2$ corresponding to different patterns of adjustment.

When $(k, l) \in A_1$, $k$ is adjusted and the firm incurs the fixed cost. In this subset, the value function satisfies:

$$v(k, l) = \sup_{k' > k, l'} \{ v(k', l') - c - (k' - k) - h(l' - l)1_{[l' \geq 0]} - f(l - l')1_{[l' < 0]} \}.$$  

This means that, starting from $(k, l) \in A_1$, the firm jumps to an optimal point $(k^*, l^*)$ satisfying the following first-order conditions:

$$\begin{cases} 
v_k(k^*, l^*) - 1 = 0, \\
v_l(k^*, l^*) - h \leq 0, \text{ with } v_l(k^*, l^*) - h = 0 \text{ when } l^* > l, \\
v_l(k^*, l^*) + f \geq 0, \text{ with } v_l(k^*, l^*) + f = 0 \text{ when } l^* < l.
\end{cases}$$  

(6)

When $(k, l)$ is not in $A_2$, only labor is adjusted, incurring a proportional cost. Then, for $(k, l) \in A_2$, the value function satisfies:

$$v(k, l) = \sup_{l'} \{ v(k, l') - h(l' - l)1_{[l' \geq 0]} - f(l - l')1_{[l' < 0]} \}.$$  

First-order conditions yield:

$$\begin{cases} 
l^* > l, \text{ with } v_l(k, l^*) - h = 0, \text{ when } v_l(k, l) > h, \text{ (case 1)}, \\
l^* = l, \text{ when } -f \leq v_l(k, l) \leq h, \text{ (case 2)}, \\
l^* < l \text{ with } v_l(k, l^*) + f = 0, \text{ when } v_l(k, l) < -f, \text{ (case 3)}.
\end{cases}$$  

(7)

Case 1 and case 3 describe situations for which $(k, l)$ is initially located in the region $A_2$: the firm instantaneously adjusts its labor force to equalize the marginal value of a job with the cost of hiring $(l^* > l)$ or firing $(l^* < l)$. Case 2 is inaction: labor or capital are unchanged, so that $(k, l) \in I$. When $(k, l)$ is in $A_2$, the firm exerts an instantaneous control on the level of $l$, while the capital stock is unchanged.
Figure 1: Inaction region and optimal return points

Figure 1 depicts an hypothetical situation. The two curves $u$ and $U$ represent the set of $(k, l)$ satisfying respectively $v_l(k, l) = h$ and $v_l(k, l) = -f$. As soon as a labor adjustment occurs, the firm proceeds instantly to a point on one of these curves. The solid part of these curves define boundaries of the inaction region. In the area between $u$ and $U$, the marginal reward to changing $L$ is insufficient: neither hiring nor firing is optimal. The curve $S$ represents the optimal return point when capital is adjusted. First-order conditions indicate that the marginal intertemporal return on capital equals one along this curve.

The dotted and numbered arrows describe different patterns of adjustment starting from the forbidden region. Adjustments 1 and 2 consider the case of a change in labor alone, with a vertical displacement to one of the curves $u$ and $U$. The starting points are located on the sub-set $A_2$. Adjustments 3, 4, 5 describe an investment for which the firm bears the fixed cost. These adjustments start in the sub-set $A_1$, and are necessarily accompanied by a jump. From points 3 and 5, the firm jumps to the optimal points $A$ or $B$. Starting from point 4, the firm do not adjust its labor force and moves horizontally toward the return curve $S$. All these changes are optimal only if the discrete increase in the value of the firm is higher than the total cost of adjustment including the fixed component. It means that we may expect that there exists a curve $s$ (solid line) which separates the inaction region from the sub-set $A_1$.

We conclude that the inaction region lies between the two curves $u$ and $U$ and at the right of the boundary $s$.

3.2 Properties of the value function in the region of inaction

Assume that the firm is located in the interior of the region inaction. When no action is undertaken, the ratio $y = k/l$ is constant and the firm evolves stochastically on a ray with a drift pushing it toward the origin. Since $y$ is time invariant in $\mathcal{I}$ it is convenient at this stage to express the value as a function of $(\pi, y)$, with $\pi = k^a l^b$.

$v(.)$ expresses the value whatever the arguments appearing in the value function.
When the firm is in its inaction region, \( v(.) \) satisfies the Bellman equation:

\[
rv(\pi, y) = \pi - w\pi^{1+s}y^a + E_t \left[ \frac{dv}{dt}(\pi, y) \right],
\]

where \( \pi \) is a geometric brownian process with drift \(-(a + b)(\delta - \sigma^2)\) and standard error \((a + b)\sigma\).

Ito’s lemma yields:

\[
rv(\pi, y) = \pi - w\pi^{1+s}y^a - \delta v_\pi(\pi, y) + \frac{1}{2} \sigma^2 (a + b)^2 \pi^2 v_{\pi\pi}(\pi, y).
\]

(8)

One associates with this differential equation a characteristic polynomial \( P(\tau) = r + (\delta + \frac{1}{2}\sigma^2) (a + b)\tau - \frac{b}{2}(a + b)^2 \alpha \tau^2 \), with roots \( \tau^+ > a + b \) and \( \tau^- < 0 \).

Moreover, a particular solution to (8) is given by:

\[
\pi / P(1) - (w\pi^{1+s}y^a) / (r + \delta).
\]

One concludes that in the region of inaction, the value function satisfies:

\[
v(\pi, y) = \frac{\pi}{P(1)} - \frac{w\pi^{1+s}y^a}{r + \delta} + T^+(y)\pi^+ + T^-(y)\pi^-,
\]

(9)

where \( T^+(.) \) and \( T^-(.) \) are the constants (w.r.t. to \( \pi \)) of integration, depending only on \( y \). As usual in a problem of control involving a brownian motion, these functions will be fully co-determined with the boundaries separating the inaction and forbidden regions.

From (9), one may deduce the marginal values of labor and capital in the region of inaction:

\[
p(\pi, y) = \frac{\pi}{P(1)} + \sum_{i = +, -} \Psi^i(\pi)\pi^\omega^i,
\]

(10)

\[
q(\pi, y) = \frac{\pi}{P(1)} + \sum_{i = +, -} \Phi^i(\pi)\pi^\omega^i,
\]

(11)

with:

\[
\Psi^i(\pi) = (yT^i(\pi) + a\tau^iT^i(\pi))y^{-\omega^i} a_{a+b}^{(1+i)} \omega^i, i = +, -
\]

\[
\Phi^i(\pi) = (-yT^i(\pi) + b\tau^iT^i(\pi))y^{-\omega^i} a_{a+b}^{(1+i)} \omega^i, i = +, -
\]

\[
\omega^i = \frac{1 - (a + b)\tau^i}{1 - a - b} i = +, -
\]

3.3 A set of boundary conditions

The functions \( T^+(.) \) and \( T^-(.) \) are determined by using a set of conditions holding on the boundary between the inaction and “forbidden” regions: the value matching conditions (VMC), the smooth pasting conditions (SPC) and the high-contact conditions (HCC). VMCs involve the value function \( v(.) \), while SPCs concern its first-order derivatives, and HCCs its second-order derivatives.

As it is emphasized by Dumas [1991], HCCs and some SPC account for the optimality of the CIIC policy, while VMCs and some SPC are deduced from the definition of the value function whatever the optimality of the combined impulse-instantaneous control policy. For all these derivations, it is easier to express the value as a function of \( (k, l) \).

\(^3\)\( f_x(.) \) denotes the first derivative of the function \( f(.) \) w.r.t the variable \( x \), while \( f_{xx}(.) \) is the second derivative w.r.t. the variable \( x \).

\(^4\) One can check that \( \omega^- < 0 < 1 < \omega^+ \) and \( (1 - b)\omega^+ > (1 - a - b)\omega^+ > 1 \).
We begin by providing the VMCs and some SPC resulting from the definition of the value function.

First, let \((k, l)\) be located in \(A_1\) (including part of the curve \(s\) in figure \(3.1\) separating \(A_1\) from \(I\)). From this \((k, l)\), it is optimal to incur the fixed cost and to buy capital, thereby causing a jump to an optimal point located on the curve \(S\) of figure 1. The increase in the value of the firm due to this adjustment equals the cost of the adjustment. The corresponding value-matching condition is:

\[
v(k, l) = v(k^*, l^*) - c - (k^* - k) - h(l^* - l)1_{\{y \geq q\}} - f(l - l')1_{\{y \geq l'\}},
\]

with \((k^*, l^*) \in S\) and \((k, l) \in A_1\). This VMC holds along the boundary \(s\) of the figure 1.

Second, let \((k, l)\) be in \(A_2\), so that only labor is adjusted toward a point of \(u\) (hiring) or \(U\) (firing) (cases 1 and 2 of the figure 1). Consider the case of hiring. From \((k, l)\), the firm moves vertically to reach a point of \(u\), by hiring \(\Delta l\) employees. One gets a value-matching condition:

\[
v(k, l) = v(k, l + \Delta l) - h\Delta l.
\]

Now, by making \(\Delta l \to 0\) and taking the derivative, we get the smooth pasting condition holding along \(u\):

\[
v_l(k, l) = h, \forall (k, l) \in u.
\]

By considering a firing policy, one gets a second SPC holding along the boundary \(U\):

\[
v_l(k, l) = -f, \forall (k, l) \in U.
\]

These two SPCs (13) and (14) boil down to the first-order conditions (7) indicating that it is optimal to exert an instantaneous control once the boundary is reached.

In addition to the conditions (12), (13) and (14), we have conditions expressing the fact that the CIIC policy undertaken by the firm is locally optimal, i.e. a control exerted whenever the firm reaches the boundaries of the inaction region yields a maximal value. These optimality conditions result from the intertemporal trade-off faced by the firm when it decides to exert a control. Indeed, along a boundary separating the inaction region \(I\) from the “forbidden” one \(A\), the firm has to be indifferent at the margin between an adjustment at date \(t\) and waiting \(dt\) to make the adjustment at date \(t + dt\), only if a shock pushing the firm in the forbidden region occurs. These marginal conditions are fully derived in the appendix A by considering the discrete approximation of a Brownian motion like a random walk.

Along the boundaries separating \(I\) from \(A\), where only adjustment of labor occurs (i.e. corresponding to the bold portion of the curves \(u\) and \(U\)), we obtain two so-called high-contact conditions:

\[
0 = (u(k) - ku'(k))^2v_u(k, u(k)), \quad (15)
\]
\[
0 = (U(k) - ku'(k))^2v_u(k, U(k)). \quad (16)
\]

\(u(.)\) and \(U(.)\) denote the two functions assumed (at least locally) to be associated with the curves \(u\) and \(U\). A prime stands for the derivative. The HCCs express that either the boundary \(u\) or \(U\) is locally impassable, either the marginal value \(v_l(.)\) remains differentiable when the boundary \(u\) or \(U\) is reached.

The boundary \(s\) has to be divided in two parts \(s_1\) and \(s_2\) depending on the return point of the firm. The part \(s_1\) corresponds to an adjustment toward the point \(A\) of the figure 3.1, while \(s_2\) is associated with an horizontal jump to a point of the curve \(S\). It is shown in the appendix A that the optimality conditions along \(s\) involve the first derivative of the value function, and as such can be seen as smooth pasting conditions. They are given by:

\[
0 = k(v_k(s(l), l) - 1) + l(v_s(s(l), l) - h), \quad \text{if} \ (k, l) \in s_1, \quad (17)
\]
\[
0 = (s(l) - ls'(l))(v_k(s(l), l) - 1), \quad \text{if} \ (k, l) \in s_2, \quad (18)
\]

with \(s(.)\) the function associated with the curve \(s\), and \(s'(.)\) its derivative w.r.t. \(l\).

\footnote{This includes the cases of a jump to point \(A\) or \(B\).}
4 Long-run properties of the combined hiring/firing and investment policy

This section contains the main result of the paper. First, we entirely characterize the ergodic set associated to the optimal policy followed by the firm. It is shown that in the long-run, “small” hiring spikes are followed by a “big” double spike of investment and hiring. Second, we derive the stationary unconditional probability density of the controlled input levels. By integrating various variables against this density, we are able to characterize aggregate variables as a function of the underlying parameters.

4.1 Combined dynamics and ergodic set

Figure 2 depicts three hypothetical trajectories in the $k - l$ plane starting from the inaction region.

Trajectory of type 1 starts on a ray crossing boundaries $u$ and $U$. Once the firm reaches the lower control barrier $u$, the optimal labor demand policy regulates the marginal value of labor never allowing it to go above $h$. This means that it hires workers to evolve just above the curve $u$ until it attains the point $C$ with coordinates $(\tilde{k}, \tilde{l})$, where a double spike of hiring and investment displaces the firm to the optimal point $A$ with coordinates $(k^*, l^*)$. This type of trajectory occurs when the ratio $y$ is below the threshold value $\tilde{y} = \tilde{k}/\tilde{l}$ lower than $y^* = k^*/l^*$. This trajectory is ergodic in the sense that, once optimal point $A$ has been reached, the trajectory of type 1 reproduces itself as a trajectory of type 1. As it is examined below, other trajectories correspond to a transient phase.

Trajectory of type 2 starts from the inaction region on a ray associated to a ratio $y$ lower than $\tilde{y}$, and crossing the lower part of the boundary $s$ and the firing frontier $U$. A firm located on this path is pushed by favorable productivity shocks toward the absorbing barrier $s$, along which a sudden double spike resets it to the return point $A$. A trajectory of type 2 is surely transformed in a finite time in a trajectory of type 1. Trajectory of type 2 is transient.

Trajectory of type 3 crosses the firing frontier $U$ and the higher part of $s$, i.e. occurs for the lowest value of the ratio $y$. A spike of investment, without hiring, takes place pushing the firm inside the inaction region on the curve $S$. A series of zigzag paths occurs until the trajectory is transformed in type 2, and ultimately in type 1.

From this informal discussion, we conjecture that there exists an ergodic set, reached in a finite time whatever the initial condition, in which the firm is trapped and is driven by a type 1 behavior of hiring, firing and investment. This set is depicted in figure 2 by the shaded area delimited by three frontiers: (i) the part of the curve $u$ originating from the absorbing point $C$, passing through point $A$ and continuing until infinity; (ii) the segment $[CD]$ with a slope $\tilde{y}^{-1}$; (iii) the part of the curve $U$ originating from point $D$ and continuing until infinity.

The next sub-section provides a formal derivation of each of these boundaries and entirely characterizes the ergodic set.

4.2 Characterizing the ergodic set

The ergodic set $\mathcal{E}$ is entirely characterized by the reflecting barriers $u$ and $U$, the absorbing point $C$ with coordinates $(\tilde{k}, \tilde{l})$ and the return point $A$ with coordinates $(k^*, l^*)$.

Assume that the firm is located in the ergodic set $\mathcal{E}$ so that $y = k/l > \tilde{k}/\tilde{l} = \tilde{y}$. Figure 2 shows that the corresponding ray is entirely located in the ergodic set and attains the two reflecting barriers $U$ and $u$. One gets the following HCCs: $v_l(u(k), k) = p_l(u(k), k) = 0$ and $v_l(U(k), k) = p_l(U(k), k) = 0$. $p(u(k), k = h$ means that $p_k(u(k), k) = 0$, and $p(U(k), k) = -f$ implies that $p_k(U(k), k) = 0$. Using
From this, it is immediate that both \((\Phi^+, \Phi^-)\) and \((\pi_l, \pi_U)\) are constants independent of \(y\) as soon as \(y > \bar{y}\).

We follow Abel and Eberly [1996] to solve this system.

Define:

\[
G = \frac{\pi_l}{\pi_U} \quad \text{and} \quad \theta(G) = \frac{G^{\omega^+} - G}{G^{\omega^+} + G^{\omega^-}}.
\]

Equations (20) and (22) yields:

\[
\omega^+ \Phi^+ P(1) = -\pi_l^{-\omega^+} (1 - \theta(G)) \tag{23}
\]

\[
\omega^- \Phi^- P(1) = -\pi_l^{-\omega^-} \theta(G) \tag{24}
\]

Then, equations (19) and (21) give:

\[
\frac{\pi_l}{P(1)} \left\{ 1 - \frac{1}{\omega^+} (1 - \theta(G)) - \frac{1}{\omega^-} \theta(G) \right\} = h + \frac{w}{r + \delta} \tag{25}
\]

\[
\frac{\pi_l}{P(1)} \left\{ G - \frac{G^{\omega^+}}{\omega^+} (1 - \theta(G)) - \frac{G^{\omega^-}}{\omega^-} \theta(G) \right\} = -f + \frac{w}{r + \delta}. \tag{26}
\]
From these two equations, one deduces the condition:
\[
J(R, G) = R - G - \frac{1}{\omega^+} (R - G^{\omega^+})(1 - \theta(G)) - \frac{1}{\omega^-} (R - G^{\omega^-})\theta(G) = 0
\]
with \( R = \frac{f + \pi}{h + \frac{\rho}{\Gamma}} \).

**Result 1** For any \( R \in ]0, 1[ \), there exists a unique \( G \in ]0, 1[ \) such that \( J(R, G) = 0 \).

**Proof** See Appendix B.

In appendix D, we derive an approximation of \( G \) for values of \( R \) close to 1. One gets:
\[
(G - 1)^3 \simeq \frac{6\sigma^2(1 - a - b)^2}{r + \delta} (R - 1) \iff G \simeq \left[ \frac{6\sigma^2(1 - a - b)^2}{r + \delta} (R - 1) \right]^\frac{1}{3} + 1
\]

If \( f < \frac{w}{1 + c} \), then, \( R \in ]0, 1[ \) and result 1 ensures that \( G \) exists. Knowing \( G \) then allows to determine \( \pi_j \) and \( \pi_f \) using equations (25) and (26) while \( \Phi^+ \) and \( \Phi^- \) are deduced from (23) and (24).

It follows that the hiring and firing boundaries are respectively given by :
\[
k = u(l) = \left( \frac{\pi_j}{bh} \right)^\frac{1}{\alpha} l^\frac{1}{a} \quad \text{and} \quad k = U(l) = \left( \frac{\pi_f}{bf} \right)^\frac{1}{\alpha} l^\frac{1}{a}.
\]

Appendix D derives the complete expression for the value function in the ergodic set:
\[
v(k, l) = \frac{k^a l^b}{P(1)} - \frac{wl}{P(0)} - \frac{1}{\omega^+} P(1)^{-1} (1 - \theta(G)) \frac{b^{\omega^+}}{(b - 1)\omega^+ + 1} k^a \omega^+ l^{b - 1}\omega^+ + 1
\]
\[
+ C^+ \frac{a^{\omega^+} k^{(a + b - 1)\omega^+ + 1}}{(a + b - 1)\omega^+ + 1} - \frac{1}{\omega^-} P(1)^{-1} \theta(G) \frac{b^{\omega^-}}{(b - 1)\omega^- + 1} k^a \omega^- l^{b - 1}\omega^- + 1
\]

with \( C^+ \) a constant to be determined.

We are now able to determine both the absorbing point \( C (\bar{k}, \bar{l}) \) and the optimal point \( A (k^*, l^*) \).
First, we know that these two points are located on the curve \( u \). Second, the optimality condition requires \( q(k^*, l^*) = 1 \) while the smooth pasting condition (17) implies \( q(\bar{k}, \bar{l}) = 1 \). Third, the value matching condition (12) links \( v(k^*, l^*) \) and \( v(\bar{k}, \bar{l}) \). It gives five equations for five unknowns \( (k^*, l^*, \bar{k}, \bar{l}, C^+) \).

Tedious algebra presented in appendix F show that \((\bar{k}, k^*)\) satisfies the following system of equations:
\[
- \frac{(b - 1)\omega^+ + 1}{a\omega^+} \Gamma \left( k^{\frac{\omega^+}{a\omega^+}} - k^{\frac{\omega^-}{a\omega^+}} \right) = (k^* - \bar{k}) + \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} - c \tag{27}
\]
\[
k^{(1 - a - b)\omega^+ - 1} \left\{ \Gamma k^{\frac{\omega^-}{a\omega^+}} - k^* \right\} = \bar{k}^{(1 - a - b)\omega^- - 1} \left\{ \Gamma \bar{k}^{\frac{\omega^-}{a\omega^+}} - \bar{k} \right\} \tag{28}
\]
where \( \Gamma \) is a positive constant defined by:
\[
\Gamma = \frac{a}{P(1)} \left( 1 - \frac{b(1 - \theta(G))}{(b - 1)\omega^+ + 1} - \frac{b\theta(G)}{(b - 1)\omega^- + 1} \right) \left( \frac{b}{\pi} \right)^\frac{1}{\alpha}
\]
It is impossible to provide an explicit solution to the system (27) (28), however we are able to prove the following result.

**Result 2** Under the condition:
\[
c < \frac{1 - a - b}{a} \frac{\frac{1}{\alpha}}{\Gamma^{\frac{1 - a - b}{a}}},
\]
there exists a unique \((k^*, \bar{k})\) with \( k^* > \bar{k} \) solution of the system (27) (28).
Proof: See appendix G One may approximate the values of $k^*$ and $\bar{k}$ for small values of $c$. If $C_f = 0$, then $k^* = \bar{k}$. One has:

$$(k^* - \bar{k})^3 = \frac{3}{2} \frac{(1 - b)\bar{k}^2}{(1 - a - b)^2\omega^+} C_f \iff k^* = \left[ \frac{3}{2} \frac{(1 - b)\bar{k}^2}{(1 - a - b)^2\omega^+} C_f \right]^\frac{1}{3} + \bar{k}
$$

and

$$(k - \bar{k})^3 = \frac{3}{2} \frac{(1 - b)\bar{k}^2}{(1 - a - b)^2\omega^+} C_f \iff k^* = -\left[ \frac{3}{2} \frac{(1 - b)\bar{k}^2}{(1 - a - b)^2\omega^+} C_f \right]^\frac{1}{3} + \bar{k}
$$

Once the ergodic set $\mathcal{E}$ has been described, it is natural to consider the long-run distribution of the variable $(k, l)$ over this set. More precisely, we are looking for a stationary distribution associated to the optimal control policy. By integrating various variables using this stationary distribution, we will be able to express aggregate variables as stationary first-order moments.

To compute the stationary density, it is useful to consider the variables $(\pi_l, y)$ instead of $(k, l)$. The density function $f(\pi_l, y)$ is defined for $\pi \in [\pi_l, \pi_u]$ and $y \in [y, \infty]$. The process is subject to two reflecting barrier. If $\pi_l$ becomes (strictly) greater than $\pi_u$, it is adjusted such that it is equal to $\pi_u$. Likewise, if $\pi_l$ becomes smaller than $\pi_l$, it is adjusted to $\pi_l$. There also are an absorbing barrier. If $y = y^*$ and $\pi_l = \pi_u$, then $y$ is resetting to $y^*$ whereas $\pi_l$ stay equal to $\pi_u$.

One has the following result about the density function of the process followed by $(\pi_l, y)$.

**Result 3** The steady state density function of the process $\{\pi_l, y_t\}$ is:

$$f(\pi_l, y) = Q_1 \pi_l^\varepsilon + Q_2 y^{-(1-b):\varepsilon} \text{ if } y \in [y, y^*)$$

with:

\begin{align*}
Q_1 &= \frac{1}{(y^* - y) \left[ \frac{\pi_l^{(1-b):\varepsilon + 1}}{\varepsilon + 1} - \frac{(\pi_l - \pi_u)^{1:0}}{-\varepsilon(1-b) + 1} \right]}
\end{align*}

$$Q_2 = -\frac{y^{(1-b):\varepsilon}}{(y^* - y) \left[ \frac{\pi_l^{(1-b):\varepsilon + 1}}{\varepsilon + 1} - \frac{(\pi_l - \pi_u)^{1:0}}{-\varepsilon(1-b) + 1} \right]}$$

and

$$f(\pi_l, y) = Q_3 y^{-(1-b):\varepsilon} \text{ if } y \in [y^*, \infty]$$

with:

\begin{align*}
Q_3 &= -\frac{(y^{(1-b):\varepsilon} - y^{* (1-b):\varepsilon}) \pi_l^{(1-b):\varepsilon + 1}}{(y^* - y) \left[ \frac{\pi_l^{(1-b):\varepsilon + 1}}{\varepsilon + 1} - \frac{(\pi_l - \pi_u)^{1:0}}{-\varepsilon(1-b) + 1} \right]}
\end{align*}

**Proof** See Appendix G The proof is simply an adaptation of Plehn-Dujowich [2005] and Dixit [1993].

By a change of variables, we can deduce the long-run distribution of $(k, l)$. We thus will be able to make numerical investigations to explore the effects of uncertainty and adjustment cost size on the long run capital and labor levels.
References


A Local optimality of the CIIC policy

This appendix provides an informal derivation of the set of boundary conditions expressing the optimality of the CIIC policy.
A.1 Discrete approximation of a geometric Brownian motion

These marginal conditions are derived by considering the discrete approximation of a Brownian motion.

Divide in discrete periods of length $\Delta t$, and assume at this stage that the stochastic process $(k, l)$ is uncontrolled. This means that, in the $(k, l)$-plane, the firm evolves along the radius emanated from the origin. We denote $\Delta x$ as being the discrete shock hitting the process $(k, l)$ during the period. The length of the period and the magnitude of the shock are related by $\Delta x = \sigma \sqrt{\Delta t}$. A “bad” (see the discussion in the text) shock is realized with a probability $p = \frac{1}{2} \left[ 1 - \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \sqrt{\Delta t} \right]$ and it moves the firm from $(k, l)$ to $(k, l) \times e^{\Delta x}$. Conversely, a “good” shock may occur with a probability $q = 1 - p$ and drives the firm towards the point $(k, l) \times e^{-\Delta x}$.

A.2 Optimality of the CIIC policy

Our aim is to characterize the boundaries of the inaction region by showing the intertemporal trade-off faced by the firm. On a boundary of the inaction region, when the optimal policy is undertaken, the firm is indifferent between two options: exercising instantly an adjustment, and bearing the cost, or postponing its decision at time $\Delta t$ waiting for the occurrence of a new shock. In what follows, we compare these two alternatives, using the discrete approximation of a geometric Brownian motion and provide conditions under which they yield the same return.

A.2.1 $(k, l) \in u$ or $U$

The firm is located on the hiring boundary $u$, and we assume, at least locally, that this boundary defines a differentiable function: $l = u(k)$. The trade-off faced by the firm is depicted in figure A.2.1. Starting from $(k, u(k))$ (point $A$) at time 0, an inactive firm would move to points $B$ (inside the region of inaction) or $C$ (outside $I$), depending on the occurrence of a bad or a good shock at time $\Delta t$. If a control policy is undertaken, the firm chooses between two alternatives.

First, the firm may decide to hire instantly $\Delta l$ workers at time 0, moving to point $A'$ for avoiding to move out of the region of inaction in case of a “good” shock at time $\Delta t$. This means that $(k, u(k) + \Delta l) \times \exp(-\Delta x)$ (point $C'$) has to be located on the boundary $u$, i.e.:

$$(u(k) + \Delta l) \exp(-\Delta x) = u(k \exp(-\Delta x)).$$

This equality defines the required amount of hiring, $\Delta l$.

The value of this action is obtained by making a second order Taylor expansion of $v(.)$ around $(k, u(k))$:

$$v(k, u(k) + \Delta l) - h\Delta l \approx v(k, u(k)) + \frac{1}{2} v_{ll}(k, u(k))(\Delta l)^2,$$

The first order term vanishes because of the equality $v_l(u(k), k) = h$. Using the equality defining $\Delta l$, one may obtain the second-order Taylor expansion of the value of an instantaneous adjustment, that is:

$$v(k, u(k) + \Delta l) - h\Delta l \approx v(k, u(k)) + \frac{1}{2} (u(k) - ku'(k))^2 v_{ll}(k, u(k))(\Delta x)^2. \quad (29)$$

Second, the firm may choose to wait $\Delta t$ before taking a decision of adjustment only if the good shock occurs. The waiting value is then:

$$R = \pi(k, l) \Delta t + \frac{pv(k e^{\Delta x}, u(k) e^{\Delta x}) + (1 - p)v(k e^{-\Delta x}, u(k) e^{-\Delta x})}{1 + r \Delta t}.$$

With a probability $1 - p$, the firm is outside the region of inaction (point $C$ of the figure A.2.1) and needs to adjust its labor force of an amount $\Delta l'$ from $u(k)e^{-\Delta x}$ to $u(k)e^{-\Delta x}$. As a result, the firm
Figure 3: HCC: \((k, l) \in u\)
The probability $p$ of the waiting value. Using the Bellman equation (8), we obtain:

$$u$$

Along the boundary $u$, the firm is indifferent between making an instantaneous adjustment at date $t$ and waiting for a time $\Delta t$. Comparing the equations (29) and (33) yields the high contact condition:

$$(u(k) - ku'(k))^2v_{ll}(k, u(k))(\Delta x)^2 = 0.$$  (31)

The case of the firing boundary $U$ is identical, and one gets the condition:

$$(U(k) - kU'(k))^2v_{ll}(k, U(k)) = 0.$$  (32)

A.3 $(k, l) \in s$

We now derive optimality condition holding on the $s$ curve. We compare the return from an instantaneous adjustment toward a point of $S$, with the value of waiting a fraction $\Delta t$, and to make the adjustment if the good shock occurs. Optimality of the boundary $s$ requires that the firm remains indifferent between these alternatives.

Figure A.3 depicts two situations depending on the occurrence of hiring going with the investment. This corresponds to the division of the curve $s$ in two parts, $s_1$ and $s_2$. If $(k, l) \in s_2$, which is the part of curve above the point $N$, then only capital is adjusted, and the firm moves horizontally toward the curve $S$. On the contrary, if $(k, l) \in s_1$, both labour and capital are adjusted, and whatever the starting point, the return point is $R^* = (k^*, l^*)$

A firm located in $A_2$ with coordinates $(s(l), l)$, can exert an impulse control (dotted arrow starting from $A_2$), yielding a value:

$$v(s(l), l) = v(S(l), l) - c - (S(l) - s(l)), $$

or it can postpone its decision, waiting for a ”good” shock (point $C_2$) or a ”bad” shock (point $B_2$), and then undertaking the control only in the ”good” shock case (dotted arrow starting from $C_2$). This postponing policy generates the return:

$$R = \pi(s(l), l)\Delta t + \frac{pv(s(l)e^{\Delta x}, le^{\Delta x}) + (1 - p)v(s(l)e^{-\Delta x}, le^{-\Delta x})}{1 + r\Delta t}.$$  (33)

Due to a value matching condition, one gets:

$$v(s(l)e^{-\Delta x}, le^{-\Delta x}) = v(s(le^{-\Delta x}, le^{-\Delta x}) - (s(le^{-\Delta x}) - e^{-\Delta x}s(l))).$$

Knowing that $p = \frac{1}{2} \left[ 1 - \left( \frac{\Delta x}{x} - \frac{1}{2} \right) \Delta x \right]$, the first-order (in terms of $\Delta x$) Taylor expansion of $R$ around $(s(l), l)$ is:

$$R \approx v(s(l), l) + \frac{1}{2} (v_k(s(l), l) - 1) \left( s(l) - ls^l(l) \right) \Delta x.$$  (34)

Comparing this quantity with $v(s(l), l)$, one deduces that the firm is indifferent between adjusting and postponing along the boundary $s_2$ when:

$$(v_k(s(l), l) - 1) \left( s(l) - ls^l(l) \right) = 0, (s(l), l) \in s_2.$$

Note that this implies that, whatever the action undertaken (adjust instantly or wait $\Delta t$), the firm reaches the point $C$ in case of a “good” shock.

As for $u$ and $U$, we use $s(.)$ and $S(.)$ as a notation for the curves $s$ and $S$ seen as function of $l$. 

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Figure 4: HCC: $(k, l) \in s$
We consider now the case for which the return is \( R^* \). A firm located in \( A_1 \) with coordinates can exert instantly an impulse control (dotted arrow starting from \( A_1 \)) to reach \( R^* \), yielding a value:

\[
v(s(l), l) = v(k^*, l^*) - c - (k^* - s(l)) - h(l^* - l),
\]
or it can postpone its decision, waiting for a "good" shock (point \( C_1 \)) or a "bad" shock (point \( B_1 \)), and then undertaking the control only in the "good" shock case (dotted arrow starting from \( C_1 \)). As before, the value of this postponing policy is given by the expression (33).

Due to a value matching condition, one gets:

\[
v(s(l)e^{-\Delta x}, l)e^{-\Delta x}) = v(s(l), l) - (s(l) - s(l)e^{-\Delta x}) - h(l - le^{-\Delta x}).
\]

A first-order expansion of \( R \) gives:

\[
R \approx v(s(l), l) + \frac{1}{2}(l(v_l(s(l), l) - h) + s(l)(v_k(s(l), l) - 1)) \Delta x.
\]

Comparing this quantity with \( v(s(l), l) \), one deduces that the firm is indifferent between adjusting and postponing along the boundary \( s_1 \) when:

\[
l(v_l(s(l), l) - h) + s(l)(v_k(s(l), l) - 1) = 0, (s(l), l) \in s_1.
\]

### B Proof of result 1

This appendix is devoted to the proof of result 1. Before doing it, we introduce some notations and useful intermediary results.

We have the following relations :

\[
\theta(G^{-1}) = G^{\omega^{-1}}\theta(G)
\]

\[
1 - \theta(G^{-1}) = G^{\omega^{-1}}(1 - \theta(G))
\]

Recall that \( P(\omega) \) is defined by :

\[
P(\omega) = -\frac{1}{2}\sigma^2(a + b - 1)^2\omega^2 - (1 - a - b)(\delta - \frac{1}{2}\sigma^2)\omega + r + \delta
\]

This polynomial has two roots satisfying : \( \omega^- < 0 < 1 < \omega^+ \).

Knowing that, we find that :

\[
\lim_{G \to O^+} \theta(G) = \lim_{G \to O^+} \frac{G^{\omega^+ - \omega^-} - G^{1-\omega^-}}{G^{\omega^+ - \omega^-} - 1} = 0
\]

Applying L’Hôpital rule, one get :

\[
\lim_{G \to 1} G^{-1} \theta(G) = \lim_{G \to 1} \frac{\omega^+G^{\omega^+ - 1}}{\omega^+G^{\omega^+ - 1} - \omega^-G^{\omega^- - 1}} = \frac{\omega^+ - 1}{\omega^+ - \omega^-} < 1
\]

Finally, one has :

\[
\lim_{G \to +\infty} \theta(G) = \lim_{G \to +\infty} \frac{1 - G^{1-\omega^+}}{1 - G^{\omega^- - \omega^+}} = 1
\]

Let define \( u = \omega^+ - 1 > 0 \) and \( v = \omega^+ - \omega^- > 0 \). \( \theta(G) \) then writes \( \theta(G) = \frac{1 - G^{v}}{1 - G^{-v}} \) and its derivative for \( G \in ]0, 1[ \cup ]1, +\infty[ \) is given by :

\[
\theta'(G) = \frac{G^{-u-v-1}}{(1 - G^{-v})^2}(u(G^v - 1) - v(G^u - 1))
\]
It is useful determine the sign of \( \theta'(G) \). Consider the function \( T(G) = u(G^v - 1) - v(G^u - 1) \). One has \( T(0) = -u + v = 1 - \omega^- > 0 \) and:

\[
\lim_{G \to +\infty} T(G) = \lim_{G \to +\infty} G^v(u - uG^{v-1} - vG^{u-v} + v) = +\infty
\]

The derivative of \( T(G) \) is given by \( T'(G) = uG^{v-1} - G^{u-1} \). It is then obvious that \( T \) reaches a minimum at \( G = 1 \). It follows that \( T(G) > 0 \) for all \( G \in [0, 1[ \cup ]1, +\infty[ \). To determine the value of \( \theta'(1) \), we apply l’Hôpital rule. One gets:

\[
\lim_{G \to 1} \frac{u(G^v - 1) - v(G^u - 1)}{(1 - G^v)^2} = \lim_{G \to 1} \frac{uv(G^{v-1} - G^{u-1})}{2vG^{v-1}(1 - G^v)} = \lim_{G \to 1} \frac{uv((v - 1)G^{v-2} - (u - 1)G^{u-2})}{2v(-v - 1)G^{v-2} - 2v(-2v - 1)G^{2v-2}} = \frac{u(-u + v)}{2v} = \frac{(1 - \omega^+)(1 - \omega^-)}{2(\omega^- - \omega^+)} = \frac{P(1)}{-\sigma^2(a + b - 1)^2(\omega^- - \omega^+)} > 0
\]

and thus, \( \theta'(1) = \frac{u(-u + v)}{2v} > 0 \). We can conclude that \( \theta'(G) > 0 \) for all \( G > 0 \).

Here, we give some useful relationships.

\[
P(\omega) = \frac{1}{2} \sigma^2 (a + b - 1)^2 (\omega^- - \omega^+)(\omega^- - \omega^-)
\]

\[
\omega^+\omega^- = \frac{r + \delta}{-\sigma^2(a + b - 1)^2}
\]

\[
P(\omega) = \frac{r + \delta}{\omega^+\omega^-}(\omega^- - \omega^+)(\omega^- - \omega^-)
\]

\[
P(1) = \frac{(1 - \omega^+)(1 - \omega^-)}{\omega^+\omega^-}
\]

\[
\frac{P(1)}{r + \delta} = \left( \frac{1}{\omega^+} - \frac{1}{\omega^-} \right) \theta(1) + \left( 1 - \frac{1}{\omega^+} \right)
\]

One also has:

\[
\theta'(1) = \frac{u(v - u)}{2v} = \frac{(1 - \omega^+)(1 - \omega^-)}{2(\omega^- - \omega^+)} = \frac{P(1)}{\sigma^2(1 - a - b)^2(\omega^+ - \omega^-)}
\]

To study the solution of the equation \( J(R, G) = 0 \), we need to give some results concerning the function \( J \) and its derivatives.

\[
J(R, 1) = R - 1 - \frac{1}{\omega^+}(R - 1)(1 - \theta(1)) - \frac{1}{\omega^+}(R - 1)\theta(1)
\]

\[
= (R - 1)\frac{P(1)}{r + \delta} \quad \forall R < 1
\]

After some algebra, the following expression of \( J(R, G) \) is obtained:

\[
J(R, G) = \frac{G}{1 - G^{\omega^+-\omega^+}} \left\{ (RG^{-1} - 1)(1 - G^{\omega^- - 1}) - \frac{1}{\omega^+}(RG^{-1} - G^{\omega^+ - 1})(G^{1 - \omega^+} - G^{\omega^- - 1}) - \frac{1}{\omega^-}(RG^{-1} - G^{\omega^+ - 1})(1 - G^{1 - \omega^+}) \right\}
\]
The term in brackets tends to \((-1 + \frac{1}{\omega^+} + 0) < 0\) as \(G\) tends to infinity. Thus:

\[
\lim_{G \to +\infty} J(R, G) = \lim_{G \to +\infty} \frac{G}{1 - G\omega^+ - \omega^-} \left(-1 + \frac{1}{\omega^+} + 0\right) = -\infty
\]

Concerning the derivatives of \(J(R, G)\), one has:

\[
J_1(R, G) = 1 - \frac{1}{\omega^+}(1 - \theta(G)) - \frac{1}{\omega^-}\theta(G)
\]

We easily deduce that:

\[
J_1(R, 1) = \frac{P(1)}{r + \delta}
\]

It also immediately follows that \(J_{11}(R, G) = 0\) and \(J_{111}(R, G) = J_{112}(R, G) = 0\).

Computing the derivative of \(J(R, G)\) with respect to \(G\) gives:

\[
J_2(R, G) = \frac{1}{G} \left[\frac{G\omega^+ - G + \theta(G)(G\omega^- - G\omega^+)}{G\omega^+ - G}\right]
+ \left[\frac{1}{\omega^+}(R - G\omega^+) - \frac{1}{\omega^-}(R - G\omega^-)\right]
\]

\(\theta(G)\) being defined by \(\theta(G) = \frac{G\omega^+ - G}{G\omega^+ - G\omega^-}\), the above expression reduces to:

\[
J_2(R, G) = \left[\frac{1}{\omega^+}(R - G\omega^+) - \frac{1}{\omega^-}(R - G\omega^-)\right]
\]

Let define:

\[
M(R, G) = \frac{1}{\omega^+}(G\omega^+ - R) - \frac{1}{\omega^-}(G\omega^- - R)
\]

The first and second derivative with respect to \(G\) are:

\[
M_2(R, G) = G\omega^+ - 1 - G\omega^- - 1
\]
\[
M_{22}(R, G) = (\omega^+ - 1)G\omega^+ - 2 - G\omega^- - 2
\]

It is obvious that:

\[
M(1, 1) = 0
\]
\[
M_2(1, 1) = 0
\]
\[
M_{22}(1, 1) = \omega^+ - \omega^-
\]
\[
J_2(1, 1) = 0
\]

The derivatives of \(J(R, G)\) can easily be rewritten as follows:

\[
J_2(R, G) = -M(R, G)\theta'(G)
\]
\[
J_{22}(R, G) = -M_2(R, G)\theta'(G) - M(R, G)\theta''(G)
\]

We also have the following cross-derivatives:

\[
J_{12}(R, G) = -M_1(R, G)\theta'(G) = \left(\frac{1}{\omega^+} - \frac{1}{\omega^-}\right)\theta'(G)
\]
\[
J_{122}(R, G) = \left(\frac{1}{\omega^+} - \frac{1}{\omega^-}\right)\theta''(G)
\]
\[
J_{222}(R, G) = -[M_{22}(R, G)\theta'(G) + 2M_2(R, G)\theta''(G) + M(R, G)\theta'''(G)]
\]
Recall that (see appendix ??) that \(\theta''(G) = \frac{(v-3-2a)\theta'(G)}{3}\) and that \(\theta''(G)\) is finished. It follows that:

\[
\begin{align*}
J_{12}(1,1) &= \left(\frac{1}{\omega^+} - \frac{1}{\omega^-}\right) \theta'(1) \\
J_{122}(1,1) &= \left(\frac{1}{\omega^+} - \frac{1}{\omega^-}\right) \theta''(1) \\
J_{222}(1,1) &= -(\omega^+ - \omega^-) \theta'(1) = -\frac{P(1)}{\sigma^2(1-\omega/\omega')} \\
\end{align*}
\]

We can now study the existence of a solution to equation \(J(R, G) = 0\) with respect to \(G\). Using the definition of \(M(R, G)\), the equation to solve can be rewritten as follows:

\[
R - G - \frac{1}{\omega^+} (R - G^\omega^+) = \theta(G) \left[ \frac{R - G^\omega^-}{\omega^-} - \frac{R - G^\omega^+}{\omega^+} \right] = \theta(G) M(R, G)
\]

or

\[
M(R, G) = \frac{1}{\theta(G)} \left[ \frac{1}{\omega^+} (G^\omega^+ - R) - (G - R) \right]
\]

Recall that \(\omega^+ > 1\) and suppose \(R < 1\). Let define:

\[
F(G) = \frac{1}{\omega^+} (G^\omega^+ - R) - (G - R)
\]

Computing the first and second derivatives of \(F(G)\) gives \(F'(G) = G^{\omega^+-1} - 1\) and \(F''(G) = (\omega^+ - 1)G^{\omega^+-2} > 0\). Thus, \(F(G)\) attains a minimum at \(G = 1\). One has \(F(1) = (1 - R)\left(\frac{1}{\omega^+} - 1\right) < 0\).

Moreover, \(F(G)\) attains the following limits:

\[
\begin{align*}
\lim_{G \to 0^+} F(G) &= R \left(1 - \frac{1}{\omega^+}\right) > 0 \\
\lim_{G \to +\infty} F(G) &= \lim_{G \to +\infty} \left\{ G^{\omega^+} \left(\frac{1}{\omega^+} - G^{1-\omega^+}\right) + R \left(1 - \frac{1}{\omega^+}\right) \right\} = +\infty
\end{align*}
\]

Or aim is to show that \(J(R, G) = \) has a unique solution with respect to \(G\) for \(R < 1\) and such that \(G \in [0, 1]\). To begin, we show the existence. Recall that \(J(R, 1) = (R - 1)\frac{P(1)}{r + \delta} < 0\). Moreover, computing the limit of \(J(R, G)\) as \(G\) tends to \(0^+\) and \(+\infty\) gives:

\[
\begin{align*}
\lim_{G \to 0^+} J(R, G) &= \lim_{G \to +\infty} \left\{ R - G - \frac{1}{\omega^+} (R - G^\omega^+) \frac{G^{1-\omega^-} - 1}{G^{\omega^+ - \omega^-} - 1} \\
&\quad - \frac{1}{\omega^-} (RG^{\omega^-} - 1)\frac{G^{\omega^+} - G}{G^{\omega^+ - \omega^-} - 1} \right\} = R \left(1 - \frac{1}{\omega^+}\right) > 0 \\
\lim_{G \to +\infty} J(R, G) &= -\infty
\end{align*}
\]

\(J(R, G)\) being an continuous function of \(G\), the existence is established, there exists \(G \in [0, 1]\) such that \(J(R, G) = 0\).

We now study the unicity. Recall that we have the following expressions:

\[
J_2(R, G) = -M(R, G)\theta'(G)
\]
We study the sign of $M(R, G)$ for $G \in ]0, +\infty[$ and $R \in ]0, 1[$. One has:

$$M(R, G) = \frac{G^\omega - R}{\omega^+} - \frac{G^\omega - R}{\omega^-}$$

$$M_2(R, G) = G^{\omega^+} - 1 - G^{\omega^-} - 1$$

It is easy to show that $\lim_{G \to 0^+} M(R, G) = +\infty$, $\lim_{G \to +\infty} M(R, G) = +\infty$ and $M(R, 1) = (1 - R) \left( \frac{1}{\omega^+} - \frac{1}{\omega^-} \right) > 0$. As $M_2(R, G) = 0$ only if $G = 1$ and $M_2(1, 1) = \omega^+ - \omega^- > 0$, $M(R, G)$ attains a unique minimum for $G = 1$. It follows that if $R < 1$, $M(R, G) > 0$ and $J_2(R, G) < 0$ for all $G > 0$.

Suppose now that $R = 1$, which occurs if $f = h = 0$. One has:

$$J(1, G) = 1 - G - \frac{1}{\omega^+}(1 - G^{\omega^+})(1 - \theta(G)) - \frac{1}{\omega^-}(1 - G^{\omega^-})\theta(G)$$

It is obvious that $J(1, 1) = 0$. We now show the unicity. We have $\lim_{G \to 0^+} J(1, G) = 1 - \frac{1}{\omega^+}$ and $\lim_{G \to +\infty} J(1, G) = -\infty$. The derivative of $J(1, G)$ with respect to $G$ writes:

$$J_2(1, G) = -M(1, G)\theta'(G)$$

We study the sign of $M(1, G)$ for $G \in ]0, +\infty[$. One has:

$$M(1, G) = \frac{G^\omega - 1}{\omega^+} - \frac{G^\omega - 1}{\omega^-}$$

$$M_2(1, G) = G^{\omega^+} - 1 - G^{\omega^-} - 1$$

$$M_2(1, G) = (\omega^+ - 1)G^{\omega^+ - 2} - (\omega^- - 1)G^{\omega^- - 2}$$

We have:

$$\lim_{G \to 0^+} M(1, G) = +\infty$$

$$\lim_{G \to +\infty} M(1, G) = +\infty$$

$M_2(1, G) = 0$ only if $G = 1$. Thus, because of the above conditions, $M(1, G)$ attains a unique minimum for $G = 1$. $M(1, G) > 0$ and $J_2(1, G) < 0$ for all $G \in ]0, 1]$, $]1, +\infty[$ and $M(1, 1) = M_2(1, G) = 0$. Consequently, $G = 1$ is the single root of $J(1, G) = 0$.

### C The derivative of $\theta(G)$

We have $\theta(G) = \frac{1}{\omega^+}G^{\omega^+ - 1} = \frac{n(G)}{d(G)}$ with $n(G) = 1 - G^{-u}$ and $d(G) = 1 - G^{-v}$. Let define $m_0(G) = n(G) = 1 - G^{-u}$, $\theta(G)$ then write $\theta(G) = \frac{m_0(G)}{d(G)}$. The derivatives of $\theta(G)$ have the following expression $\theta(j)(G) = \frac{m_j(G)}{d(G)}$. Differentiating this expression provides:

$$\theta^{(j+1)}(G) = \frac{m'_j(G)d(G) - m_j(G)d'(G)}{d(G)^2} = \frac{m_{j+1}(G)}{d(G)}$$

Thus, one has $m_{j+1}(G) = m'_j(G) - \theta(j)(G)d'(G)$. This expression will be used to determine the values of $\theta(G)$ and $\theta^{(j)}(G)$ for $G = 1$.

We have $m_0(1) = d(1) = 0$. Thus, applying l’Hôpital rule provides $\theta(1) = \frac{m_0'(1)}{d'(1)} = \frac{u}{v}$. It follows that $m_1(1) = m_0'(1) - \theta(1)d'(1) = 0$ and $\theta'(1) = \frac{m_1'(1)}{d'(1)}$ by application of l’Hôpital rule. Successive iterations finally give $m_j(1) = m_{j-1}'(1) - \theta^{(j-1)}(1)d'(1) = 0$ and $\theta^{(j)}(1) = \frac{m_j'(1)}{d'(1)}$.

We can now compute $\theta'(1)$, $\theta''(1)$...

To compute $\theta'(1)$, consider $m_1(G) = m_0(G) - \theta(G)d'(G)$ and differentiate it. One gets:

$$m_1'(G) = m_0''(G) - \theta(G)d''(G) - \theta'(G)d'(G)$$

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It follows that:

\[ m'_1(1) = m''_0(1) - \theta(1)d''(1) - \theta'(1)d'(1) = \theta'(1)d'(1) \]

Solving with respect to \( \theta'(1) \) provides:

\[ \theta'(1) = \frac{m''_0(1) - \theta(1)d''(1)}{2d'(1)} = \frac{u(v - u)}{2v} \]

To compute \( \theta''(1) \), consider \( m_2(G) = m'_1(G) - \theta'(G)d'(G) \) and \( m'_1(G) = m''_0(G) - \theta(G)d''(G) - \theta'(G)d'(G) \). Differentiating these expressions with respect to \( G \) provides:

\begin{align*}
    m'_2(G) &= m''_1(G) - \theta''(G)d'(G) - \theta'(G)d''(G) \\
    m''_1(G) &= m'''_0(G) - \theta'(G)d'''(G) - \theta''(G)d''(G) - \theta'(G)d'(G)
\end{align*}

Substituting the expression of \( m''_0(G) \) in the one giving \( m'_2(G) \) provides:

\[ m'_2(G) = m'''_0(G) - 3\theta'(G)d''(G) - \theta(G)d'''(G) - 2\theta''(G)d'(G) - \theta''(G)d''(G) \]

It follows that:

\[ m'_2(1) = m'''_0(1) - 3\theta'(1)d''(1) - \theta(1)d'''(1) - 2\theta''(1)d'(1) = \theta''(1)d'(1) \]

Solving with respect to \( \theta''(1) \) gives:

\[ \theta''(1) = \frac{m'''_0(1) - 3\theta'(1)d''(1) - \theta(1)d'''(1)}{3d'(1)} \]

\( m'''_0(1), d''(1) \) and \( d'''(1) \) are easy to compute and \( \theta(1) \) and \( \theta'(1) \) was previously determined. After some algebra, one gets:

\[ \theta''(1) = \frac{(v - 3 - 2U)\theta'(1)}{3} \]

Applying the same method, a tedious algebra would allow to determine the value of \( \theta'''(1) \) which is finished. We omit this algebra because we only need to know that \( \theta'''(1) \) is finished.

\section{D Determination of the approximate value of \( G \)}

If \( R = 1, J(1, G) = 0 \) for \( G = 1 \). Make a three order development of \( J(R, G) = 0 \) to obtain:

\begin{align*}
    J(R, G) &= J(1, 1) + J_1 (1, 1)(R - 1) + J_2(G - 1) + \frac{1}{2} [J_{11}(1, 1)(R - 1)^2 \\
    &+ 2J_{12}(1, 1)(R - 1)(G - 1) + J_{22}(1, 1)(G - 1)^2] \\
    &+ \frac{1}{6} [J_{111}(1, 1)(R - 1)^3 + 3J_{112}(1, 1)(R - 1)^2(G - 1) \\
    &+ 3J_{122}(1, 1)(R - 1)(G - 1)^2 + J_{222}(1, 1)(G - 1)^3] \]
\end{align*}

Take the values of the derivatives of \( J(R, G) \) for \( R = 1 \) and \( G = 1 \) determined at appendix ?? and rearrange the terms to obtain:

\[ (G - 1)^3 = \frac{6\sigma^2(1-a-b)^2}{r+\delta}(R - 1) \]
E Implicit prices of capital and labor and the value function in the ergodic set

Using the result of the paper, one may deduce that for all value of \((k, l)\) such that \(k/l > y\), the marginal value of labor is

\[
p(k, l) = \frac{bk^a l^{b-1}}{P(1)} - \frac{w}{r + \delta} - \frac{1}{\omega^+} \frac{\partial}{\partial l} \left[ \frac{1}{\omega^+} (1 - \theta(G)(bk^a l^{b-1})) \omega^+ - \frac{1}{\omega^-} \frac{\partial}{\partial l} \theta(G)(bk^a l^{b-1}) \omega^- \right]
\]

Moreover, from equation (10), the implicit price of capital is given by:

\[
q(k, l) = \frac{ak^{a-1}l^b}{P(1)} + \Psi^+ \left( \frac{k}{l} \right) (ak^{a-1}l^b) \omega^+ + \Psi^- \left( \frac{k}{l} \right) (ak^{a-1}l^b) \omega^-
\]

We now determine the unknown functions \(\Psi^+(.)\) and \(\Psi^-(.)\) for all values of \((k, l)\) such that \(k/l > y\).

We exploit the restriction \(p_k = p_l\) to obtain a set of differential equations:

\[
\begin{align*}
\frac{k}{l} \Psi^+ \left( \frac{k}{l} \right) + b\omega^+ \Psi^+ \left( \frac{k}{l} \right) &= -\frac{1}{\omega^+} \frac{\partial}{\partial l} (1 - \theta(G))a \left( \frac{b}{a} \right) \omega^+ \left( \frac{k}{l} \right)^{\omega^+ - 1} + C^+ \left( \frac{k}{l} \right)^{b\omega^+} \\
\frac{k}{l} \Psi^- \left( \frac{k}{l} \right) + b\omega^- \Psi^- \left( \frac{k}{l} \right) &= -\frac{1}{\omega^-} \frac{\partial}{\partial l} \theta(G) a \left( \frac{b}{a} \right) \omega^- \left( \frac{k}{l} \right)^{\omega^- - 1} + C^- \left( \frac{k}{l} \right)^{b\omega^-}
\end{align*}
\]

Solve these differential equations gives:

\[
\begin{align*}
\Psi^+ \left( \frac{k}{l} \right) &= -\frac{1}{\omega^+} \frac{\partial}{\partial l} (1 - \theta(G)) a \left( \frac{b}{a} \right) \omega^+ \left( \frac{k}{l} \right)^{\omega^+ - 1} + C^+ \left( \frac{k}{l} \right)^{b\omega^+} \\
\Psi^- \left( \frac{k}{l} \right) &= -\frac{1}{\omega^-} \frac{\partial}{\partial l} \theta(G) a \left( \frac{b}{a} \right) \omega^- \left( \frac{k}{l} \right)^{\omega^- - 1} + C^- \left( \frac{k}{l} \right)^{b\omega^-}
\end{align*}
\]

which implies:

\[
q(k, l) = \frac{ak^{a-1}l^b}{P(1)} - \frac{1}{\omega^+} \frac{\partial}{\partial l} (1 - \theta(G)) a \left( \frac{b}{a} \right) \omega^+ \left( \frac{k}{l} \right)^{\omega^+ - 1} + C^+ \left( \frac{k}{l} \right)^{b\omega^+} \\
+ C^+ a^{\omega^+} k^{(a+b-1)\omega^+} - \frac{1}{\omega^-} \frac{\partial}{\partial l} \theta(G) a \left( \frac{b}{a} \right) \omega^- \left( \frac{k}{l} \right)^{\omega^- - 1} + C^- \left( \frac{k}{l} \right)^{b\omega^-} \\
+ C^- a^{\omega^-} k^{(a+b-1)\omega^-}
\]

Integrate \(q\) with respect to \(k\) for the value function for all \((k, l)\) with \(k/l > y\):

\[
v(k, l) = \frac{k^a l^b}{P(1)} - \frac{w l}{P(0)} - \frac{1}{\omega^+} \frac{\partial}{\partial l} (1 - \theta(G)) a \left( \frac{b}{a} \right) \omega^+ \left( \frac{k}{l} \right)^{\omega^+ - 1} + C^+ \left( \frac{k}{l} \right)^{b\omega^+} \\
+ C^+ a^{\omega^+} k^{(a+b-1)\omega^+} - \frac{1}{\omega^-} \frac{\partial}{\partial l} \theta(G) a \left( \frac{b}{a} \right) \omega^- \left( \frac{k}{l} \right)^{\omega^- - 1} + C^- \left( \frac{k}{l} \right)^{b\omega^-} \\
+ C^- a^{\omega^-} k^{(a+b-1)\omega^-}
\]

The constant \(C^-\) is necessarily equal to 0. Indeed, recall that no adjustments on capital are made if \(k \to +\infty\). In that case, \(l\) is adjusted in the long run along the boundary \(U(k)\) and 

\[
k = \left( \frac{\nu}{\beta} \right)^{\frac{1}{\gamma - b}} l^{\frac{1}{\gamma - b}}
\]

or 

\[
l = \left( \frac{\nu}{\beta} \right)^{\frac{1}{\gamma - b}} k^{\frac{1}{\gamma - b}}
\]

It follows that the term in \(C^-\) in the expression of the implicit price of capital take an infinite value. We thus necessarily have \(C^- = 0\).
F Determination of the system giving \( k^* \) and \( \tilde{k} \)

To sum up, one has the following set of equations:

\[
q(k, L) = \frac{a_k}{P(1)} - \frac{a_k}{\theta(G)} (1 - \theta(G)) \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + C + a_k \omega^+ \frac{k_a + b - 1}{a_k + b - 1} \Omega^+ \frac{a_k}{(b-1)\omega^+ + 1} \Omega^- \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + \frac{a_k}{\theta(G)} (1 - \theta(G)) \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + C + a_k \omega^+ \frac{k_a + b - 1}{a_k + b - 1} \Omega^+ \frac{a_k}{(b-1)\omega^+ + 1} \Omega^- \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} = 1
\]

\[
q(k^*, L^*) = \frac{a_k}{P(1)} + \frac{a_k}{\theta(G)} (1 - \theta(G)) \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + C + a_k \omega^+ \frac{k_a + b - 1}{a_k + b - 1} \Omega^+ \frac{a_k}{(b-1)\omega^+ + 1} \Omega^- \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} = 1
\]

\[
v(k^*, L^*) - v(k, L) = \frac{1}{P(1)} \left( k_a + b - 1 \right) + \frac{a_k}{\theta(G)} (1 - \theta(G)) \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + C + a_k \omega^+ \frac{k_a + b - 1}{a_k + b - 1} \Omega^+ \frac{a_k}{(b-1)\omega^+ + 1} \Omega^- \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} = h(l^* - l) + (k^* - \tilde{k}) + c
\]

Knowing that \( l^* = \frac{b}{a} k^{a+b} \), and after elimination of the constant \( C^+ \) and a tedious algebra, the above system rewrite as follows:

\[
\begin{align*}
-k^* + \frac{1}{P(1)} \left( (b-1)\omega^+ + 1 \right) & \frac{a_k}{\theta(G)} (1 - \theta(G)) \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} + C + a_k \omega^+ \frac{k_a + b - 1}{a_k + b - 1} \Omega^+ \frac{a_k}{(b-1)\omega^+ + 1} \Omega^- \frac{a_k}{(b-1)\omega^+ + 1} \frac{a_k}{L} = h(l^* - l) + (k^* - \tilde{k}) + c
\end{align*}
\]

G Proof of result 2

Let define the function:

\[
\Omega(x) = - \frac{(b-1)\omega^+ + 1}{a\omega^+} \Gamma x^{\frac{a}{\omega^+}} - x
\]

\( \Omega(x) \) satisfies the following properties.

\[
G'(x) = \frac{(b-1)\omega^+ + 1}{(b-1)\omega^+} \Gamma x^{\frac{a}{\omega^+}} - 1
\]

\[
G''(x) = \frac{a + b - 1}{1 - b} \frac{(b-1)\omega^+ + 1}{(b-1)\omega^+} \Gamma x^{\frac{a}{\omega^+}} < 0
\]

\[
G(0) = 0
\]

\[
\lim_{x \to +\infty} G(x) = -\infty
\]

The function \( \Omega(x) \) has a unique maximum at \( x = \tilde{k} \equiv \left( (b-1)\omega^+ + 1 \right) \left( \frac{1}{1 - a - b} \right) \) and one has \( \Omega(x) = 0 \) for \( x = \tilde{k} \equiv \left( (b-1)\omega^+ + 1 \right) \left( \frac{1}{1 - a - b} \right) \). The following lemma is satisfied:
Lemma 1 If \( C_F \) satisfies the following inequality:

\[
\frac{(a + b - 1)\omega^+}{(a + b - 1)\omega^+ + 1} \frac{1 - a - b}{a} \left( \frac{(b - 1)\omega^+ + 1}{(b - 1)\omega^+} \right)^{1 - \frac{b}{a - b}} \geq C_F
\]

then, there exists \( k_0^- > 0 \) and \( k_0^+ > 0 \) such that \( k_0^- < \tilde{k} < k_0^+ \) and satisfying \( G(k_0^-) = G(k_0^+) = \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \).

Consider the equation \( G(x) = \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \). This equation has solutions if and only if \( \arg \max_x G(x) = G(\tilde{k}) \geq \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \), that is:

\[
\frac{(a + b - 1)\omega^+}{(a + b - 1)\omega^+ + 1} \frac{1 - a - b}{a} \left( \frac{(b - 1)\omega^+ + 1}{(b - 1)\omega^+} \right)^{1 - \frac{b}{a - b}} \geq C_F
\]

If the above inequality is strict, one has two distinct roots \( k_0^+ \) and \( k_0^- \) satisfying \( k_0^+ > \tilde{k} > k_0^- \) and \( G'(k_0^+) > 0 \) and \( G'(k_0^-) < 0 \). If one has equality, there is one (double) root such that \( k_0^+ = k_0^- = \tilde{k} \).

Now, the solution of equation \( G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \) for each \( x \geq 0 \) is studied. Before studying its solutions, one gives some useful properties about the function \( G(x) \). This is the object of the two following corollaries.

Corollary 1 1. If \( x < k_0^- \), then \( G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F < 0 \) and \( G'(x) > 0 \);

2. If \( k_0^- < x < \tilde{k} \), then \( G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F > 0 \) and \( G'(x) > 0 \);

3. If \( \tilde{k} < x < k_0^+ \), then \( G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F > 0 \) and \( G'(x) < 0 \);

4. If \( x > k_0^+ \), then \( G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F < 0 \) and \( G'(x) < 0 \).

Corollary 2 1. If \( x < \tilde{k} \), then \( G(x) > 0 \) and \( G'(x) > 0 \);

2. If \( x = \tilde{k} \), then \( G(x) > 0 \) and \( G'(x) = 0 \);

3. If \( \tilde{k} < x < k_0^- \), then \( G(x) > 0 \) and \( G'(x) < 0 \);

4. If \( x = k_0^- \), then \( G(x) = 0 \) and \( G'(x) < 0 \);

5. If \( x > k_0^- \), then \( G(x) < 0 \) and \( G'(x) < 0 \).

Consider the equation \( G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \). The following lemma (immediately) follows:

Lemma 2 1. If \( x < k_0^- \), then, \( \exists y > \tilde{k} \) such that \( G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \). More over, one has \( G'(y) < 0 \) and \( x < y \);

2. If \( x = k_0^- \), then, \( G(k_0^-) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F = 0 \) and there exists \( y^- = 0 \) and \( y^+ = \tilde{k} \) such that \( G(y^-) = 0 \) and \( G(y^+) = 0 \). Moreover, \( G'(y^-) > 0 \) and \( G'(y^+) < 0 \);

3. If \( k_0^- < x < \tilde{k} \), then, there exists \( y^- < \tilde{k} \) and \( y^+ \in [\tilde{k}, k_0^-] \) such that \( G(y^-) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \), \( G'(y^-) > 0 \) and \( G'(y^+) < 0 \). Moreover, \( x < y^+ \) and \( y^- < x \);

4. If \( \tilde{k} < x < k_0^+ \), there exists \( y^- < \tilde{k} \) and \( y^+ \in [\tilde{k}, k_0^+] \) such that \( G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_F \), \( G'(y^-) > 0 \) and \( G'(y^+) < 0 \). Moreover, \( y^- < x \) and \( x < y^+ \).
5. If \( x > k_0^+ \), then, \( \exists ! y > \tilde{k} \) such that \( G(y) = G(x) - \frac{(a+b-1)\omega^+ + 1}{(a+b-1)\omega^+} C_F \), \( G'(y) < 0 \) and \( x < y \).

Let now define the :

\[
\Upsilon(x) = x^{(1-a-b)\omega^+} \left( \Gamma_x \frac{a+b}{1-b} - 1 \right) = x^{(1-a-b)\omega^+ - 1} \left( \Gamma_x \frac{a}{1-b} - 1 \right)
\]

This function satisfy the following properties. \( F(x) = 0 \) if and only if \( x = \tilde{k} \equiv \Gamma^\frac{1-b}{1-a-b} \) or \( x = 0 \). The derivative of \( F(x) \) writes :

\[
F'(x) = (1-a-b)\omega^+ k^{(1-a-b)\omega^+ - 1} \left[ \frac{(1-b)\omega^+ - 1}{(1-b)\omega^+} \right]^{\Gamma_k \frac{a}{1-a-b} - 1 - 1}
\]

One easily checks that \( F'(0) = +\infty \) if \( (1-a-b)\omega^+ - 1 + \frac{a}{1-b} - 1 < 0 \) and \( F'(0) = 0 \) if \( (1-a-b)\omega^+ - 1 + \frac{a}{1-b} - 1 > 0 \). It follows that \( F'(x) = 0 \) if and only if \( x = \tilde{k} \) or \( x = 0 \) if \( (1-a-b)\omega^+ - 1 + \frac{a}{1-b} - 1 > 0 \).

One also has \( \lim_{x \to +\infty} F(x) = -\infty \) and \( F(\tilde{k}) = \left( \frac{(1-b)\omega^+ - 1}{(1-b)\omega^+} \right) \left( \frac{(1-b)\omega^+ - 1}{(1-b)\omega^+} \right) > 0 \). Thus, on the interval \([0, +\infty[\), the function \( F(x) \) attains a maximum at \( x = \tilde{k} \) and its first derivative satisfies \( F'(x) > 0 \) for \( x \in [0, \tilde{k}[ \) and \( F'(x) < 0 \) for \( x \in ]\tilde{k}, +\infty[ \).

One now study the solutions with respect to \( y \) and for each \( x > 0 \) of the equation \( F(y) = F(x) \). One has an obvious solution, that is \( y = x \). We study if there is an other solution. This is the object of the following lemma.

**Lemma 3**  
1. If \( x \in [0, \tilde{k}[ \), then, there exists \( y^- \in [0, \tilde{k}[ \) and \( y^+ \in ]\tilde{k}, \tilde{k}[ \) such that \( F(y) = F(x) \), \( y^- = x \), \( F'(y^-) > 0 \) and \( F'(y^+) < 0 \);
2. If \( x = \tilde{k} \), then \( F(y) = F(\tilde{k}) \) if and only if \( y = x = \tilde{k} \);
3. If \( x \in ]\tilde{k}, \tilde{k}[ \), there exists \( y^- \in [0, \tilde{k}[ \) and \( y^+ \in ]\tilde{k}, \tilde{k}[ \) such that \( F(y) = F(x) \), \( y^+ = x \), \( F'(y^-) > 0 \) and \( F'(y^+) < 0 \);
4. If \( x = \tilde{k} \), then \( F(y) = F(\tilde{k}) \) if and only if \( y = y^- = 0 \) or \( y = y^+ = \tilde{k} \);
5. If \( x > \tilde{k} \), then \( \exists ! y \) such that \( F(y) = F(x) \) and \( y = x \).

**Lemma 4** The inequality \( \tilde{k} < k_0^+ \) holds if and only if :

\[
\frac{1-a-b}{a} < C_F
\]

**Proof** \( k_0^+ \) and \( \tilde{k} \) respectively satisfies \( G(k_0^+) = \frac{(a+b-1)\omega^+ + 1}{(a+b-1)\omega^+} C_F \) and \( F(\tilde{k}) = 0 \). We know that \( k_0^+ > \tilde{k} \) and \( \tilde{k} > \tilde{k} \). Furthermore, \( \forall x > \tilde{k}, G'(k) < 0 \). It follows that \( \tilde{k} < k_0^+ \) if and only if \( G(\tilde{k}) = G(k_0^+) \). Which implies the above inequality.

**Lemma 5** The following inequality holds :

\[
\frac{(1-a-b)\omega^+}{(1-a-b)\omega^+ - 1} \left( \frac{(1-b)\omega^+ - 1}{(1-b)\omega^+} \right)^{\frac{1-b}{1-a-b}} > 1
\]
Proof Consider the function:

$$\vartheta(x) = \left(\frac{\alpha x}{\alpha x - 1}\right)^x = e^{x \ln \left(\frac{\alpha x}{\alpha x - 1}\right)}$$

with $\alpha > 1$ and $x \in \left[\frac{b}{\alpha}, 1\right]$.

Calculate the derivative of $\vartheta(x)$ to obtain:

$$\vartheta'(x) = \left[\ln \left(\frac{\alpha x}{\alpha x - 1}\right) - \frac{1}{\alpha x - 1}\right] \vartheta(x)$$

Let now define:

$$\chi(x) = \ln \left(\frac{\alpha x}{\alpha x - 1}\right) - \frac{1}{\alpha x - 1}$$

One has:

$$\chi'(x) = \frac{1}{x(\alpha x - 1)^2} > 0$$

and

$$\chi(1) = \ln \left(\frac{\alpha}{\alpha - 1}\right) - \frac{1}{\alpha - 1} < 0$$

Indeed, consider $t(\alpha) = \ln \left(\frac{\alpha}{\alpha - 1}\right) - \frac{1}{\alpha - 1}$ with $\alpha > 1$. One has $t'(\alpha) = \frac{1}{(\alpha - 1)^2} > 0$ and $\lim_{\alpha \to \infty} t(\alpha) = 0$. Necessarily, $t(\alpha)$ satisfies $t(\alpha) < 0, \forall \alpha > 1$. It follows that $\chi(1) < 0$.

Finally, $\chi(1) < 0$ and $\chi'(x) > 0, \forall x \in \left[\frac{b}{\alpha}, 1\right]$ implies that $\chi(x) < 0, \forall x \in \left[\frac{b}{\alpha}, 1\right]$. We conclude that $\vartheta'(x) < 0$. We now apply this result to the case $\alpha = \omega^+ > 1$. Consider:

$$\vartheta(x) = \left(\frac{\omega^+ x}{\omega^+ x - 1}\right)^x$$

Knowing that $1 - a - b < 1 - b$, one has $\vartheta(1 - a - b) > \vartheta(1 - b)$, which implies:

$$\frac{(1 - a - b)\omega^+}{(1 - a - b)\omega^+ - 1} \left(\frac{(1 - b)\omega^+ - 1}{(1 - b)\omega^+}\right)^{\frac{1 - a - b}{1 - a - b}} > 1$$

Consider the system:

$$G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_f$$

$$F(y) = F(x)$$

Suppose that $x \in \left[\tilde{k}, k_0^+\right]$, we know that the equation $G(y) = G(x) - \frac{(a + b - 1)\omega^+ + 1}{(a + b - 1)\omega^+} C_f$ has two roots. For each $x$, one has $y^-(x) < \tilde{k}$ and $y^+(x) \in [\tilde{k}, k]$. Let define $\zeta^+(x) = y^+(x)$. Apply the implicit function theorem to obtain $\zeta''^+(x) = G'(x)/G'(\zeta''^+(x)) < 0$. Thus, on the interval $[\tilde{k}, k_0^+]$, $y^-(x)$ is a decreasing function with $\zeta^- = \tilde{k}$ and $\tau^- = k_0^+$.

Suppose that $x \in [\tilde{k}, \tilde{k}]$, we know that the equation $F(y) = F(x)$ has two roots. For each $x$, one has $y^-(x) \in [0, \tilde{k}]$ and $y^+(x) \in [\tilde{k}, k]$ with $y^-(x) = x$. Let define $y^+(x) \equiv \eta^+(x)$. Apply the implicit function theorem to obtain $\eta''^+(x) = F'(x)/F'(\eta''^+(x)) < 0$. Thus, on the interval $[\tilde{k}, \tilde{k}]$, $\eta^-(x)$ is a decreasing function with $\eta^- = \tilde{k}$ and $\eta^- = \tilde{k}$.

Knowing that if the inequality $\frac{1 - a - b}{a} \Gamma^{\frac{1 - a - b}{1 - a - b}} > C_f$ holds, one has $\tilde{k} < k_0^+$, the function $\eta^-(x)$ and $\zeta^-(x)$ necessarily intersect on the interval $[\tilde{k}, k]$. The point of intersection is unique. The uniqueness
is ensured if \( \left| F'(k^*) \right| > \left| \frac{G'(k^*)}{G'(k)} \right| \). Knowing that \( F'(k^*) < 0, F'(k) > 0, G'(k^*) < 0 \) and \( G'(k) > 0 \), it is possible to rewrite the preceding condition as follows:

\[
- \frac{F'(k^*)}{F'(k)} > - \frac{G'(k^*)}{G'(k)} \iff \frac{F'(k^*)}{G'(k^*)} > \frac{F'(k)}{G'(k)}
\]

The above condition reduces to \( k^*(1-a-b) \omega^+ > k(1-a-b) \omega^+ \) and \( (1-a-b) \omega^+ - 1 > 0 \). Knowing that \( k^* > k \) and \( (1-a-b) \omega^+ - 1 > 0 \), the preceding inequality is necessarily satisfied. The uniqueness is thus proved.

### H Determination of the approximate values of \( k^* \) and \( k \)

Here, we compute an approximate value of \( k^* \) and \( k \) for small values of \( C_F \), knowing that if \( C_F = 0 \), then \( k^* = k = \tilde{k} \).

The system of equation giving \( k^* \) and \( k \) is:

\[
G(k^*) - G(k) = \frac{(a+b-1) \omega^+ + 1}{(a+b-1) \omega^+} C_F
\]

\[
F(k^*) = F(k)
\]

The second equation defines implicitly \( k = f(k^*) \). The first equation then rewrites \( G(k^*) - G(f(k^*)) = \frac{(a+b-1) \omega^+ + 1}{(a+b-1) \omega^+} C_F \). A three order development of the right hand side at the neighborhood of \( \tilde{k} \) provides:

\[
G(k^*) - G(f(k^*)) \approx G(\tilde{k}) - G(f(\tilde{k})) + (G'(\tilde{k}) - f'(\tilde{k})G'(f(\tilde{k}))) (k^* - \tilde{k})
\]

\[
+ \frac{1}{2} (G''(\tilde{k}) - f''(\tilde{k})G''(f(\tilde{k}))) (k^* - \tilde{k})^2
\]

\[
+ \frac{1}{6} \left( G'''(\tilde{k}) - 3 f''(\tilde{k}) f'(\tilde{k}) G''(f(\tilde{k})) - f'(\tilde{k}) G'''(f(\tilde{k})) \right) (k^* - \tilde{k})^3
\]

(36)

We now compute the terms of the above development. We first need to evaluate the derivatives of \( f(x) \) when \( x = \tilde{k} \). Consider \( F(x) = F(f(x)) \). Applying the implicit function theorem provides:

\[
f'(x) = \frac{F'(x)}{F'(f(x))}
\]

To evaluate this expression at \( x = \tilde{k} \), we apply l’Hôpital rule. Knowing that \( \tilde{k} = f(\tilde{k}) \), one gets:

\[
f'(\tilde{k}) = \frac{F''(\tilde{k})}{f'(\tilde{k}) F''(\tilde{k})} = \frac{1}{f'(\tilde{k})}
\]

It follows that \( f'(\tilde{k}) = 1 \) or \( f(\tilde{k}) = -1 \).

The second derivative of \( f(x) \) writes:

\[
f''(x) = \frac{F''(x) - f''(x) f'(x) F''(f(x))}{F'(f(x))}
\]

To evaluate this expression at \( x = \tilde{k} \), we once again apply l’Hôpital rule. One gets:

\[
f''(\tilde{k}) = \frac{F'''(\tilde{k}) - 2 f''(\tilde{k}) f'(\tilde{k}) F''(f(\tilde{k})) - f'(\tilde{k})^3 F'''(f(\tilde{k}))}{f'(\tilde{k}) F''(f(\tilde{k}))}
\]

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If \( f'(\tilde{k}) = -1 \), then:

\[
f''(\tilde{k}) = \frac{F'''(\tilde{k}) + 2f''(\tilde{k})F''(\tilde{k}) + F'''''(\tilde{k})}{-F''(\tilde{k})}
\]

Solving we respect to \( f''(\tilde{k}) \) provides: \( f''(\tilde{k}) = -\frac{2}{3} \frac{F'''(\tilde{k})}{F''(\tilde{k})} \). The expression \( 38 \) reduces to:

\[
G(k) - G(f(k)) \simeq \frac{1}{3} \left( G'''(\tilde{k}) - \frac{F'''(\tilde{k})}{F''(\tilde{k})} G''(\tilde{k}) \right) (k^* - \tilde{k})^3
\]

An easy, but tedious algebra finally provides:

\[
G(k) - G(f(k)) \simeq -\frac{2}{3} (1 - a - b) \omega^+ - 1 \frac{a + b - 1}{1 - b} \frac{1}{k^2} (k^* - \tilde{k})^3
\]

We conclude that \((k^* - \tilde{k})^3 = \frac{3}{2} \frac{(1-b)^2}{(1-a-b)^2 \omega^+} C_f \).

A similar calculus applies to determine the approximate value of \( \tilde{k} \).

**I Proof of result 3**

To obtain the stationary distribution, we use a forward Kolmogorov equation derived as follows. To begin, consider a point \((\pi, y) \in [\pi_l, \pi_u] \times [y_l, y_u] \). No adjustment are made, so, only \( \pi \) evolve. \( \pi \) can moved from \( \pi_l e^{-(1-a-b)\Delta x} \) with a probability \( p = \frac{1}{2} \left[ 1 - \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] \) (“bad” shock) or from \( \pi_l e^{-(1-a-b)\Delta x} \) with a probability \( q = 1 - p = \frac{1}{2} \left[ 1 + \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] \) (“good” shock).

We thus have:

\[
f(\pi, y) = \frac{1}{2} \left[ 1 - \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] f(\pi_l e^{-(1-a-b)\Delta x}, y) + \frac{1}{2} \left[ 1 + \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] f(\pi_l e^{-(1-a-b)\Delta x}, y)
\]

Now, take the limit as \( \Delta x \to 0 \) of the second-order Taylor expansion of the above expression. We get the following Kolmogorov equation:

\[
\left[ 1 - a - b - \frac{2}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \right] \pi_1 f_1(\pi_l, y) + (1 - a - b)\pi_1^2 f_{11}(\pi_l, y) = 0 \quad (37)
\]

The general solution of the above equation is:

\[
f(\pi_l, y) = g(y)\pi_l^\varepsilon + h(y)
\]

with \( \varepsilon = \frac{2}{\sigma^2} \frac{\delta - \frac{1}{2} \sigma^2}{1 - a - b} \). \( h(y) \) and \( g(y) \) are functions to be determined later.

Now, consider a point \((\pi, y) \) with \( y \in [y_l, y_u] \). This point can be attained from \( (\pi_l e^{-(1-a-b)\Delta x}, y) \) or from \( (\pi_l, ye^{-(1-a-b)\Delta x}) \) with a probability \( p = \frac{1}{2} \left[ 1 - \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] \) (the point \((\pi, y) \) can only be attained in case of bad shock). We thus have:

\[
f(\pi, y) = \frac{1}{2} \left[ 1 - \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] \left\{ f(\pi_l e^{-(1-a-b)\Delta x}, y) + f(\pi_l, ye^{-(1-a-b)\Delta x}) \right\} \quad (38)
\]
Now, take the limit as $\Delta x \rightarrow 0$ of the first-order Taylor expansion of the above expression. We get the following Kolmogorov equation:

$$-rac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) f(\pi_l, y) - \frac{1}{2} \left( \frac{1-a-b}{1-b} \right) y f_2(\pi_l, y) + \frac{1}{2} \pi_l f_1(\pi_l, y) = 0 \quad (39)$$

Similarly, if we consider a point on the other barrier, that is $(\pi_l, y)$, we derive the equation:

$$-rac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) f(\pi_l, y) - \frac{1}{2} \left( \frac{1-a-b}{1-b} \right) y f_2(\pi_l, y) + \frac{1}{2} \pi_l f_1(\pi_l, y) = 0 \quad (40)$$

Consider now the solution of equation (37) and substitute in equations (39) and (40). One gets:

$$\frac{1}{2} \left( \frac{1-a-b}{1-b} \right) \pi_l y g'(y) + \frac{1}{2} \left( \delta - \frac{1}{2} \sigma^2 \right) h(y) + \frac{1}{2} \frac{1-a-b}{1-b} y h'(y) = 0$$

It immediately follows that $g'(y) = 0$, thus, $g(y) = Q_1$. Solving the differential equation in $h(y)$ provides:

$$h(y) = Q_2 y - \frac{\frac{1}{2} \left( \delta - \frac{1}{2} \sigma^2 \right)}{\frac{1}{2} \left( \frac{1-a-b}{1-b} \right)} = Q_2 y^{-(1-b)}$$

Suppose that the process is at the point $(\pi_l, y^\ast)$. This point can be attained in three ways, that is $(\pi_l, y^\ast e^{-\frac{1-a-b}{1-b} \Delta x}, y^\ast)$, $(\pi_l, y^\ast e^{\frac{1-a-b}{1-b} \Delta x})$, and $(\pi_l, y^\ast e^{-\frac{1-a-b}{1-b} \Delta x} (\pi_l, y^\ast)).$ This point can only be attained after a good shock occurring with probability $1 - p = \frac{1}{2} \left[ 1 + \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right]$. We thus have:

$$f(\pi_l, y^\ast) = \frac{1}{2} \left[ 1 + \frac{1}{\sigma^2} \left( \delta - \frac{1}{2} \sigma^2 \right) \Delta x \right] \times \left\{ f(\pi_l e^{-\frac{1-a-b}{1-b} \Delta x}, y^\ast) + f(\pi_l, y^\ast e^{\frac{1-a-b}{1-b} \Delta x}) \right\} \quad (41)$$

Take the limit as $\Delta x \rightarrow 0$ to obtain:

$$f(\pi_l, y^\ast) = f(\pi_l, y^\ast) + f(\pi_l, y)$$

Finally, take the limit as $\Delta x \rightarrow 0$ of the first-order Taylor expansion of equation (41). The following expression is obtained:

$$\frac{1}{2} \left( \frac{1-a-b}{1-b} \right) \left\{ f(\pi_l, y^\ast) + f^+(\pi_l, y^\ast) + f(\pi_l, y) \right\} + \frac{1}{2} \left\{ - (1-a-b) \pi_l f_1(\pi_l, y) \right\}$$

Integrating the above differential equation and using the boundaries conditions provide the expression of result [5].